Black holes in loop quantum gravity: the complete space-time

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We consider the quantization of the complete extension of the Schwarzschild space-time using spherically symmetric loop quantum gravity. We find an exact solution corresponding to the semi-classical theory. The singularity is eliminated but the space-time still contains a horizon. Although the solution is known partially numerically and therefore a proper global analysis is not possible, a global structure akin to a singularity-free Reissner–Nordström space-time including a Cauchy horizon is suggested.

arXiv:0805.1187v1 [gr-qc] 8 May 2008

Black holes are among the most spectacular revolutions in our understanding of the nature of space-time that occurred as a consequence of general relativity. In the classical theory, certain configurations of matter cannot overcome their gravitational self attraction and form an event horizon, a surface beyond which no communication with the exterior is possible. Matter continues to contract inside the horizon until a singularity is formed. Such singularities in the theory have the desirable property of not being able to communicate with the exterior (cosmic censorship). On the other hand, it is expected that such singular behavior of the classical theory could be altered significantly when one considers quantum effects. In loop quantum gravity, for example, it is known in the context of mini-superspace models that the big bang singularity is eliminated and replaced by a bounce [1]. Since the interior of a black hole is classically isometric to a Kantowski–Sachs cosmology (that also sees its singularity eliminated in loop quantum cosmology), it is natural to expect that the black hole singularity will also disappear in a similar way [2, 3]. A complete treatment of the space-time of a black hole in loop quantum gravity is still lacking, even within a midi-superspace type of quantization. The intention of this letter is to provide such a treatment. We will consider space-times with spherical symmetry and set up their canonical theory. We will use a further gauge fixing to avoid the hard problem of having structure functions in the constraint algebra (see [4] for a good discussion). We will then proceed to study classically the "polymerized" theory that can be straightforwardly quantized in the loop representation. It is known that such polymerized theories can capture many effects that one would find in a more systematic quantization followed by a semi-classical approximation. We will see that indeed the complete space-time can be covered and a solution can be constructed that replaces the singularities (black and white hole) of the usual Kruskal diagram by regular surfaces. We will show that in fact such surfaces can be smoothly matched so where one expected a "black hole" one tunnels into a "white hole" region of another universe and this can be continued indefinitely. The resulting solution therefore has a Cauchy horizon and can be characterized as the analog in semiclassical loop quantum gravity of an eternal black hole.

We will use the Ashtekar new variables to describe the spherically symmetric space-times. Previous work on this subject was done in modern language by Bojowald and Swiderski [5] so we refer the reader to them for details. There is only one non-trivial spatial direction (the radial) which we call x since it is not necessarily parameterized by the usual radial coordinate. We will elaborate more on the range of x later. The canonical variables usual in loop quantum gravity are a set of triads E^a_i and $SO(3)$ connections A^i_a; after the imposition of spherical symmetry one is left with three pairs of canonical variables $(\eta, P^\eta, A_\varphi, E^\varphi, A_x, E^x)$. The variables θ and φ are angles transverse to the radial direction as in usual polar coordinates. Instead of using triads in the transverse directions, one introduces a "polar" set of variables E^φ, η and their canonical momenta. It is convenient to introduce the gauge invariant variable K_x defined by $2\gamma K_x = A_x + \eta'$ and also K_φ defined as $A_\varphi = 2\gamma K_\varphi$, where γ is the Immirzi parameter of loop quantum gravity. . The canonically conjugate pairs are now E^x, K_x and E^φ, K_φ. The relationship to more traditional metric variables is,

$$g_{xx} = \frac{(E^\varphi)^2}{|E^x|}, \qquad g_{\theta\theta} = |E^x|, \tag{1}$$

$$K_{xx} = -K_x \text{sign}(E^x) \frac{(E^\varphi)^2}{\sqrt{|E^x|}}, \quad K_{\theta\theta} = -\sqrt{|E^x|} K_\varphi,$$

and the latter two are the components of the extrinsic curvature. The diffeomorphism and Hamiltonian constraints can be seen in detail in ref. [6]. These constraints have the usual constraint algebra for gravity in $1 + 1$ dimensions, which includes structure functions. This implies the usual "problem of dynamics" of canonical quantum gravity. Our strategy to treat this model will be to bring it down to a model with one Abelian constraint and a true Hamiltonian. That way it can be treated using the standard Dirac procedure. To achieve this we eliminate the diffeomorphism constraint by choosing a gauge that determines the functional form for $E^x = f(x, t)$. Imposing the constraint strongly determines K_x. This also fixes the corresponding Lagrange multiplier (the shift) $N^r = -\dot{f}(x, t)/f'(x, t)$ and also breaks reparametrization invariance. One is left with a theory with a single con-

straint that is Abelian and with a true Hamiltonian, the dynamical variables are E^φ and K_φ and the constraint is, constraint,

$$\Phi = -\sqrt{E^x} - K_\varphi^2\sqrt{E^x} + \frac{1}{4}\frac{\left((E^x)'\right)^2\sqrt{E^x}}{(E^\varphi)^2} + 2M \quad (2)$$

with M an integration constant and the evolution is given by a true Hamiltonian,

$$H_{\rm true} = \int dx \frac{\dot{f}(x,t)}{f'(x,t)} E^\varphi \left(K_\varphi\right)', \quad (3)$$

which preserves the constraint upon evolution. We assume the spatial manifold (the radial direction) has two boundaries. The theory at the boundary can be constructed in similar fashion as in the exterior case so we refer the reader to [6] for reasons of space. One ends up with one degree of freedom in the boundary (the mass) that does not evolve in time and coincides with the constant M.

The quantization of the Abelian constraint is straightforward and can be carried out in the same Hilbert space we considered in the exterior case. In brief, one discretizes the radial direction and the Hilbert space is a tensor products of a Hilbert space of loop quantum cosmology per spatial point. In such a space the constraint (2) is not well defined, but one can work with an expression where K_φ is replaced by $\sin(\mu K_\varphi)/\mu$. The latter is immediately expressible in terms of holonomies and therefore naturally exists in the loop representation. The resulting theory agrees with general relativity in the limit $\mu \to 0$. In loop quantum gravity it is natural to consider a finite value of μ, usually associated with the elementary quantum of area [1].

Instead of quantizing the theory and then studying the semiclassical limit, we will follow a procedure that is known [3] to capture some of the semiclassical behaviors, in particular the elimination of the singularity, at least in simple examples with constant value of μ as the one we are considering. We analyze the resulting classical "polymerized" theory with finite μ. One then considers a classical theory of gravitation, different from general relativity that contains some of the ingredients of the quantum theory, akin to when one works out in an effective theory.

We wish to choose the function $f(t,x)$ in such a way that in the limit $\mu \to 0$ one recovers the standard Schwarzschild metric in Kruskal-like coordinates. That is, a metric with a singularity at $x^2 - t^2 = -1$. On the other hand, in the case of finite μ we would like the surface $x^2 - t^2 = -1$ to correspond to a regular surface of the metric beyond which the metric can be extended. To be more specific we will choose $E^x = f(u,t,\delta)$, where $u = x^2 - t^2 + 1$ and $\delta(\mu)$ a positive parameter such that when $\mu \to 0$, $\delta \to 0$ and we recover the standard Kruskal form of the Schwarzschild space-time. To completely fix the gauge and obtain an explicit solution we set $K_\varphi = g(u,t,\delta)$ after polymerization. In the quantum theory such a gauge fixing would be equivalent to the study of an evolving constant [7, 8] E^φ in terms of c-number variable K_φ.

We will require the following conditions on the gauge fixing. We choose the u in the range $[0,\infty]$ and is such that the radial variable has a logarithmic dependence on u, $r = \sqrt{E^x} \sim M\ln(u)$ for $u \to \infty$. Moreover, asymptotically $E^\varphi \sim r + M$ in ordinary Schwarzschild coordinates, which appropriately transformed is $E^\varphi \sim (2M/\sqrt(u))(M\ln(u) + M)$. The conjugate variables are exponentially small in the radial coordinate $K_x \sim K_\varphi \sim 1/\sqrt{u}$. These boundary conditions are very similar to those in Kruskal coordinates [9]. We did not choose to work exactly in Kruskal coordinates asymptotically given the complicated relation between r and u in those coordinates. At $u = 0$ we will require that all variables be t-independent and we will choose their derivatives to vanish (in the case of K_φ we choose the derivative of $\sin(\mu K_\varphi)$ to vanish, since it is the relevant expression for the determination of the metric components via the constraints. This ensures that one can easily continue the manifold without shells of matter present at $u = 0$. There might be other possibilities for this boundary condition but we have not explored them. Finally we would like that in the limit $\delta \to 0$ we get a gauge choice that covers the entire extension of the Schwarzschild space-time (as we mentioned, it will not be exactly the same as the Kruskal extension, but related to it via non-singular, yet complicated, coordinate transformations). Although the choice of coordinates we are making is not unique, it is computationally laborious to actually find a coordinate system that satisfies all the conditions we listed and that involves variables that do not turn complex in certain regions and that has the variable K_φ taking correct values in the Bohr compactification.

The specific choice we make for E^x is,

$$\begin{aligned} E^x &= \left\{\frac{\left[\delta\left(1+u\right) + \left(10u^2 + u^{7/2}\right)\left(\delta\left(t^2-1\right)+1\right)\right]}{u^{7/2} + (t^2-1)\left(\delta u^{7/2} + \delta^2\right) + \frac{1}{2}\delta^2 u} \right. \\ &\quad \left. \times\left[\ln(1+u)\right]^2 + \delta^8\right\} M^2. \end{aligned} \quad (4)$$

This choice has the property that for $u \to 0$ $E^x = M^2\delta^8$ independent of t, for large values of u it behaves as $M^2\left(\ln^2(u) + \delta^8\right) + O(u^{-1})$, in the limit $\delta \to 0$ we have that $E^x = M^2(10u^{3/2} + u^3)\ln^2(1+u)/u^3$ tends to 0 when $u = 0$, as in the Kruskal coordinates, giving rise to the singularity. It can be checked that the first derivative with respect to x of E^x vanishes for $u = 0$ for any finite value of δ. This choice for E^x is not unique, in the sense that other choices may satisfy the above conditions. It might be possible to find simpler choices.

For K_φ we choose,

$$K_\varphi = \frac{1}{2}\frac{\delta^{5/2}\pi\left(1+\ln\left(1+u^2\right)\right)}{\mu\left(\delta^{5/2}+\ln\left(1+u\right)^2\right)} + \frac{|t|\ln\left(1+u^3\right)}{u^{3/8}} \times \frac{\left(-1+\frac{u}{(10+u\ln(1+u))}+\frac{(1+8)u}{\left(100+u\ln(1+u)^2\right)}\right)}{\left(\delta^2 t+\ln\left(1+u^3\right)\right)\left(1+u^{1/8}\right)}. \quad (5)$$

This choice has the property that for $u \to 0$ $K_\varphi = \pi/(2\mu)$ independent of t, so the term that appears in the Hamiltonian goes as $\sin\left(\mu K_\varphi\right) \sim 1$. This means that the departure of the polymerized theory from continuum general relativity is maximum at the point where the singularity would have occurred in the continuum theory. Therefore loop quantum gravity could remove the classical Schwarzschild singularity. In the limit $\delta \to 0$ we have that K_φ blows up when $u = 0$, as in the Kruskal coordinates, also compatible with the presence of the singularity in the continuum theory. For large values of u K_φ behaves as $t/\sqrt{u} + O(u^{-1/8})$, It can be checked that the first derivative with respect to x of $\sin(\mu K_\varphi)$ vanishes for $u = 0$. As in the case of E^x, the choice is not unique. It should also be noted that the choice is only valid in $|t| > 1$. We have extended the solution beyond that domain. The extension is symmetric under $t \to -t, x \to -x$, but it makes the expressions too lengthy, so for reasons of space here we concentrate in the region $|t| > 1$ since it includes the singularity.

We would now like to generate a solution to the constraint and the evolution equations of the polymerized theory. We will adopt the following strategy: we solve the diffeomorphism constraint for K_x and the remaining constraint for E^φ. The preservation of the gauge conditions in time determine the lapse and shift. The consistency of the system, that is, the preservation of the constraints upon the Hamiltonian evolution guarantees that the evolution equations for the canonical variables are automatically satisfied.

We start by obtaining E^φ from the Hamiltonian constraint, which is immediate since the relation is algebraic,

$$E^\varphi = \frac{1}{2}\left(E^x\right)'\left(\sqrt{1-\frac{2M}{\sqrt{E^x}}+\frac{\sin\left(\mu K_\varphi\right)^2}{\mu^2}}\right)^{-1}, \quad (6)$$

recall that $(E^x)'$ is given by (4). In these expressions prime means derivative with respect to x.

Since $(E^x)'$ vanishes for $u = 0$ and one wishes E^φ to be finite there to avoid having a singularity, one needs the denominator of (6) to vanish. This condition determines the relation between μ and δ, $\mu = \frac{\delta^2}{\sqrt{2-\delta^4}}$.

It is worthwhile showing explicitly the behavior of E^φ as $u \to 0$, $E^\varphi|_{u\to 0} = 2M^2\delta^{11/16} + \left(\delta^{1/8}\left(120(t^2-1)-1\right)+120\right)M^2u^2\delta^{9/16}$ which confirms that $(E^\varphi)' = 0$ at $u = 0$, one of the conditions we wanted.

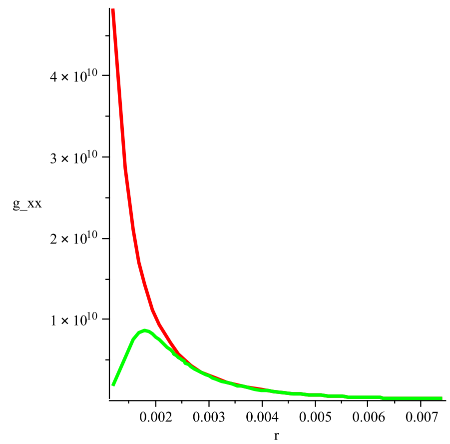

FIG. 1: The metric component g_{xx} shown as a function of r for the usual Schwarzschild solution and the solution of the polymerized theory. The plots are for $\delta = 10^{-42}$ nd $M = 1$. One sees the two graphs coincide long before one reaches the horizon at $r = 2M$, but inside the Schwarzschild solution tends to blow up at $r = 0$ whereas the solution of the polymerized theory becomes finite. Close to the origin the plotting program cannot capture that the solution of the polymerized theory grows again and takes a large, but finite, value. Although the comparison with Schwarzschild is suggestive, one must exercise care since we are plotting a coordinate dependent quantity in two different theories. One can show that both curves will agree in the limit $\delta \to 0$. The behavior of components of the curvature tensor grows monotonically as $r \to 0$ up to a maximum a finite value at the tunneling in the polymerized theory.

The behavior at large u for the metric is given by,

$$g_{xx}|_{u\to\infty} = 4\frac{M^2}{u} + 8\frac{M^2}{u\ln(u)}, \quad (7)$$

and this is just the coordinate transform of $(1 + 2M/r)$ with $r = M\ln(u)$ to leading orders in asymptotic powers of u, yielding the familiar form of the Schwarzschild solution to leading order in $1/r$.

We now proceed to determine the lapse, using the conservation in time of the second gauge condition, the one involving K_φ. We compute $\dot{K}_\varphi$ using the total Hamiltonian. From there one immediately gets

$$N' = -\frac{1}{4}\frac{\dot{K}_\varphi\left(E^x\right)' - K'_\varphi\dot{E}^x}{\left(1-\frac{2M}{\sqrt{E^x}}+\frac{\sin(\mu K_\varphi)^2}{\mu^2}\right)^{3/2}} \quad (8)$$

and via a quadrature one obtains N. We were not able to compute the expression for the latter in closed form, but numerical evaluations are straightforward. It is good to get asymptotic expressions for the integrals to use as boundary data for the numerical integrals. We can therefore reconstruct all components of the space-time metric. We can use this to study the causal structure of light cones. With this we can locate the horizon by studying

at each value of t the radial position at which the hypersurface tangent to $\sqrt{E^x} =$ const. becomes null. We have carried out the numerical computations for values $t = [10, 100]$. For larger values the computation becomes harder due to numerical issues. We will choose the parameter $\delta = 10^{-8}$ and 10^{-42} to study convergence. To understand the meaning of these values it is worthwhile noticing that the ratio between the radius at the point where the curvature takes its maximum value and the Schwarzschild radius to be of the order of $\delta^{1/14}$ Since we are dealing with the classical polymerized theory there is no notion of Planck mass. Using estimates based on the treatment of the interior using the Kantowski–Sachs isometry and that the polymerized theory departs from general relativity in scales associated with the Planck length, one can conclude that one would be dealing with a black hole of $3 - 1000$ Planck masses for both choices of δ we make. Corrections with respect to the usual Schwarzschild solution at the position of the event horizon are of the order $\delta^{1/2}$, i.e. for the choices we make from 10^{-4} to too small to be detected with the accuracy we are working.

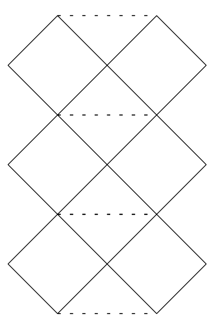

FIG. 2: The conjectured global structure of the solution. The singularity is replaced by a regular region indicated with a dashed line. The space-time is continued through into another copy of the same solution. The solution would have a Cauchy horizon similar to that in a Reissner–Nordstöm solution, presumably unstable.

Summarizing, we have carried out a midi-superspace treatment of spherically symmetric space-times in loop quantum gravity. We have studied a classical solution that captures the features of the semiclassical theory. The singularity is avoided and a picture is suggested in which the spacetime of a (highly idealized) eternal black hole is continued into another region containing a Cauchy horizon, similar to a Reissner–Nordström space-time but without the singularity. In spite of the lack of singularity, there still is a horizon and a causal behavior far away from the singularity similar to that of the usual Schwarzschild solution.

Is the solution unique? At this point we cannot say. There clearly are parameters that can be changed, but it is not clear if they just correspond to diffeomorphisms. Although the treatment of the exterior carried out previously [6] yields a single solution up to diffeomorphisms, it is known that in the treatments of the interior the “polymerization” breaks Birkhoff’s theorem [2, 3] suggesting it may not hold in the complete case either. In the interior treatment there appears an additional parameter in the solution which, for instance, controls if the “bounce” is symmetric or not and the extent of the region where the polymerized theory departs from general relativity. Our solution appears to have several free parameters, even though we have imposed by hand that the bounce be symmetric. Clarifying the uniqueness point may shed light on the degrees of freedom that are remnant of the elimination of the singularity in loop quantum gravity and may yield a picture with elements in common with the “fuzzballs” [10] of string theory, although our solutions do not exhibit significant departures from general relativity at the position of the horizon.

We wish to thank Abhay Ashtekar and Rafael Porto for comments and Miguel Campiglia for help with early calculations. This work was supported in part by grants nsf-phy0650715, and by funds of the Horace C. Hearne Jr. Institute for Theoretical Physics, FQXi, PEDECIBA and PDT #63/076 (Uruguay) and CCT-LSU.

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